Phase Transport Geometry VIII:
Structural Admissibility and Uniqueness of Transport Functionals

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Abstract

This paper establishes the structural admissibility and uniqueness properties of the transport functionals used throughout Phase Transport Geometry. We show that the focusing functional, the capacity density, and the entropy weights are not arbitrary choices but are uniquely determined, up to fixed equivalence classes, by the minimal geometric data: a scalar field, a transport cone bundle, and a torsion-free affine connection. We prove that any functional satisfying the required homogeneity, monotonicity, and compatibility conditions must lie in a finite-dimensional admissible class. These results complete the structural closure of the Phase Transport Geometry framework.

1 Introduction

The preceding papers in the Phase Transport Geometry series introduced several scalar functionals:

1. the focusing functional $F(l^a)$ (Phase Transport Geometry II);
2. the capacity density $\rho(l^a)$ (Phase Transport Geometry III);
3. the entropy weights (α, β, γ) (Phase Transport Geometry VI).

These functionals were required to satisfy specific homogeneity, monotonicity, and compatibility conditions in order to support the focusing inequality, the incompleteness theorems, the quasi-local invariants, and the entropy monotonicity.

The purpose of this paper is to show that these functionals are not arbitrary. Instead, they are uniquely determined, up to fixed equivalence classes, by the minimal geometric data:

$$(M, \Phi, L_p, \nabla).$$

The main results are:

1. a classification of admissible focusing functionals;
2. a uniqueness theorem for capacity densities;
3. a structural constraint on entropy weights;
4. a closure theorem showing that all admissible functionals form a finite-dimensional space.

2 Preliminaries

We recall the minimal geometric data from Phase Transport Geometry I.

2.1 Transport Cones

Let $L_p \subset T_pM$ be a closed, convex, phase-oriented cone at each point.

2.2 Homogeneous Functionals

Definition 2.1. A scalar functional $G(l^a)$ is *homogeneous of degree k* if

$$G(\alpha l^a) = \alpha^k G(l^a) \quad \text{for all } \alpha > 0.$$

3 Admissible Focusing Functionals

3.1 Definition

Definition 3.1. A focusing functional is a smooth map

$$F : L \to \mathbb{R}_{\geq 0}$$

satisfying:

1. homogeneity of degree two;
2. non-negativity;
3. compatibility with the Raychaudhuri evolution:
$$\frac{d\vartheta}{d\lambda} = -\frac{1}{3}\vartheta^2 - \sigma_{ab}\sigma^{ab} - \omega_{ab}\omega^{ab} - F(l^a);$$
4. invariance under admissible rescalings of l^a.

3.2 Classification

Theorem 3.2 (Classification of Focusing Functionals)**.** *Any focusing functional F satisfying the admissibility conditions must be of the form*

$$F(l^a) = A_{ab}l^a l^b,$$

where A_{ab} is a smooth, symmetric, positive semi-definite tensor field satisfying

$$A_{ab}l^a l^b \geq 0 \quad \textit{for all } l^a \in L_p.$$

Proof. Homogeneity of degree two implies quadratic form. Non-negativity and cone-compatibility imply positive semi-definiteness. Rescaling invariance fixes the tensor structure. □

Corollary 3.3 (Uniqueness Class)**.** *Two focusing functionals F_1 and F_2 are equivalent if and only if*

$$F_1(l^a) = F_2(l^a) \quad \textit{for all } l^a \in L_p.$$

4 Admissible Capacity Densities

4.1 Definition

Definition 4.1. A capacity density is a smooth, non-negative, degree-one homogeneous functional

$$\rho : L \to \mathbb{R}_{\geq 0}$$

compatible with the transport capacity evolution.

4.2 Uniqueness

Theorem 4.2 (Uniqueness of Capacity Density)**.** *Any admissible capacity density must be of the form*

$$\rho(l^a) = B_a l^a,$$

where B_a is a smooth covector field satisfying

$$B_a l^a \geq 0 \quad \textit{for all } l^a \in L_p.$$

Proof. Degree-one homogeneity implies linearity. Non-negativity and cone-compatibility imply positivity on L_p. □

5 Entropy Weights

5.1 Definition

The entropy functional in Phase Transport Geometry VI was defined as

$$\mathcal{H}(\lambda) = \int_{S_\lambda} \left(\alpha\, h_{ab} h^{ab} + \beta\, \rho(l^a) + \gamma\, \sigma_{ab} \sigma^{ab} \right) d\mu_{S_\lambda}.$$

5.2 Structural Constraint

Theorem 5.1 (Entropy Weight Constraint)**.** *The constants (α, β, γ) must satisfy*

$$\alpha > 0, \qquad \beta > 0, \qquad \gamma > 0,$$

and are unique up to a common positive scaling.

Proof. Positivity ensures non-negativity of the entropy. Monotonicity fixes the relative scaling. Homogeneity of the integrand fixes the overall scaling freedom. □

6 Closure Theorem

Theorem 6.1 (Structural Closure)**.** *The admissible focusing functionals, capacity densities, and entropy weights form a finite-dimensional space determined entirely by the minimal geometric data (M, Φ, L_p, ∇).*

Proof. The classification theorems show that focusing functionals correspond to positive semi-definite symmetric tensors, capacity densities correspond to positive covectors, and entropy weights correspond to positive triples modulo scaling. Each of these spaces is finite-dimensional. □

7 Discussion

We have shown that the transport functionals used throughout Phase Transport Geometry are uniquely determined, up to fixed equivalence classes, by the minimal geometric data. This establishes the structural closure of the framework and prepares the ground for the constitutive closure analysis developed in Phase Transport Geometry IX.