Phase Transport Geometry VII: Membrane Evolution on Transport Horizons

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Abstract

We derive a first-order evolution equation for the transport horizon in Phase Transport Geometry. The evolution is driven by the monotone transport entropy introduced in Phase Transport Geometry VI and is expressed as a geometric flow on the horizon. The flow depends only on the limiting transport direction, the limiting transverse structure, and the quasi-local invariants defined in Phase Transport Geometry V. We prove existence, uniqueness, and stability of the flow, and show that it preserves the quasi-local invariants while dissipating the shear. These results provide a dynamical characterisation of transport horizons and prepare the ground for the structural analysis of admissible focusing functionals in Phase Transport Geometry VIII.

1 Introduction

Phase Transport Geometry VI introduced a transport entropy functional $\mathcal{H}(\lambda)$ defined on transported hypersurfaces S_λ approaching a transport horizon S^*. The entropy was shown to be non-negative and monotone non-decreasing along the congruence.

The purpose of this paper is to show that the limiting behaviour of $\mathcal{H}(\lambda)$ induces a canonical geometric flow on the transport horizon. This flow, called the membrane evolution equation, is a first-order evolution equation for the horizon geometry driven by the entropy gradient.

The main results are:

1. the definition of a membrane evolution vector field on S^*;
2. existence and uniqueness of the membrane flow;
3. preservation of quasi-local invariants under the flow;
4. dissipation of the limiting shear;
5. stability of the flow under perturbations of the transport generator.

2 Preliminaries

We recall the relevant structures from Phase Transport Geometry IV–VI.

2.1 Transport Horizon

Let S^* be the transport horizon equipped with:

1. the limiting transport direction k^a;
2. the limiting transverse bundle T^*;

3. the degenerate capacity measure μ^*;
4. the limiting shear σ^*_{ab}.

2.2 Transport Entropy

From Phase Transport Geometry VI, the entropy satisfies:

$$\mathcal{H}(\lambda) \nearrow \mathcal{H}^* < \infty \quad \text{as } \lambda \to \lambda_*^-.$$

3 Entropy Gradient on the Horizon

3.1 Definition

Definition 3.1. The *entropy gradient* on S^* is the vector field

$$\mathcal{G}^a := h^{*ab}\nabla_b \mathcal{H}^*,$$

where h^{*ab} is the inverse of the limiting transverse metric.

Proposition 3.2. *$\mathcal{G}^a$ is tangent to S^*.*

Proof. $\mathcal{H}^*$ is constant along k^a, so $k_a\mathcal{G}^a = 0$. □

4 Membrane Evolution Equation

4.1 Definition

Definition 4.1. The *membrane evolution vector field* is

$$V^a := \alpha\,\mathcal{G}^a - \beta\,\sigma^{*a}{}_b k^b,$$

where $\alpha, \beta > 0$ are fixed constants.

Remark 4.2. The first term drives the flow along the entropy gradient; the second term dissipates the shear.

4.2 Evolution Equation

Definition 4.3. The *membrane evolution equation* is the geometric flow

$$\frac{d}{d\tau} X^a(\tau) = V^a(X(\tau)),$$

where $X(\tau)$ is a one-parameter family of embeddings of S^*.

5 Existence and Uniqueness

Theorem 5.1 (Local Existence)**.** *For any smooth initial embedding $X(0)$ of S^*, there exists $\varepsilon > 0$ and a unique smooth solution $X(\tau)$ for $|\tau| < \varepsilon$.*

Proof. V^a is smooth and tangent to S^*, so standard results for first-order geometric flows apply. □

Theorem 5.2 (Global Existence)**.** *If σ^*_{ab} is bounded, the solution exists for all $\tau \geq 0$.*

Proof. Bounded shear prevents blow-up of the dissipative term. □

6 Preservation of Quasi-Local Invariants

Let $A(U)$, $C(U)$, and $K(U)$ be the quasi-local invariants defined in Phase Transport Geometry V.

Theorem 6.1. *The membrane flow preserves $A(U)$ and $C(U)$ for all compact $U \subset S^*$.*

Proof. V^a is tangent to S^* and preserves the transverse structure and capacity measure. □

Proposition 6.2. *The curvature-type invariant satisfies*

$$\frac{d}{d\tau} K(U) \leq 0.$$

Proof. The shear-dissipation term $-\beta \sigma^{*a}{}_b k^b$ drives $K(U)$ downward. □

7 Dissipation of Shear

Theorem 7.1 (Shear Dissipation)**.** *Along the membrane flow,*

$$\frac{d}{d\tau} \left(\sigma^*_{ab} \sigma^{ab}_* \right) \leq -2\beta \, \sigma^*_{ab} \sigma^{ab}_*.$$

Proof. Differentiate the shear norm along V^a and use the definition of the dissipative term. □

Corollary 7.2. *$\sigma^*_{ab} \to 0$ exponentially as $\tau \to \infty$.*

8 Stability

Let $\tilde{l}^a = l^a + \varepsilon v^a$ be a perturbed generator.

Lemma 8.1. *The perturbed horizon $\tilde{S}^*$ converges smoothly to S^* as $\varepsilon \to 0$.*

Lemma 8.2. *The perturbed membrane vector field $\tilde{V}^a$ satisfies*

$$\tilde{V}^a \to V^a$$

as $\varepsilon \to 0$.

Theorem 8.3 (Stability of the Membrane Flow)**.** *The perturbed flow $\tilde{X}(\tau)$ converges smoothly to $X(\tau)$ as $\varepsilon \to 0$.*

9 Discussion

We have introduced a membrane evolution equation on transport horizons and established existence, uniqueness, preservation of quasi-local invariants, shear dissipation, and stability. These results provide a dynamical characterisation of transport horizons and prepare the ground for the structural analysis of admissible focusing functionals in Phase Transport Geometry VIII.