Phase Transport Geometry VI: Transport Entropy and Monotonicity

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Abstract

We introduce a transport entropy functional associated with transport horizons in Phase Transport Geometry. The entropy is constructed from the quasi-local invariants developed in Phase Transport Geometry V and is defined on transported hypersurfaces approaching the horizon. We prove that the entropy is well-defined, non-negative, and monotone non-decreasing along admissible transport congruences. The monotonicity is derived from the focusing inequality and the structural properties of the quasi-local invariants. These results provide the analytical foundation for the membrane evolution equation developed in Phase Transport Geometry VII.

1 Introduction

Phase Transport Geometry V introduced three quasi-local invariants associated with transport horizons: a transverse area invariant, a capacity-weighted invariant, and a curvature-type invariant derived from the limiting shear. The purpose of this paper is to combine these invariants into a single scalar functional, the transport entropy, and to establish its monotonicity along transported hypersurfaces.

The main results are:

1. the definition of a transport entropy functional on transported hypersurfaces;
2. well-definedness and non-negativity of the entropy;
3. a monotonicity theorem derived from the focusing inequality;
4. stability of the entropy under perturbations of the transport generator.

2 Preliminaries

We recall the relevant structures from Phase Transport Geometry I–V.

2.1 Transported Hypersurfaces

Let S_λ be the transported hypersurfaces generated by a twist-free transport generator l^a with $F(l^a) \geq 0$ and $\vartheta|_S < 0$.

2.2 Quasi-Local Invariants

Let $A(U)$, $C(U)$, and $K(U)$ denote the quasi-local invariants defined on compact subsets $U \subset S^*$ of the transport horizon.

3 Transport Entropy

3.1 Definition

Definition 3.1. Let S_λ be a transported hypersurface. The *transport entropy* of S_λ is

$$\mathcal{H}(\lambda) := \int_{S_\lambda} \Big(\alpha\, h_{ab}h^{ab} + \beta\, \rho(l^a) + \gamma\, \sigma_{ab}\sigma^{ab}\Big)\, d\mu_{S_\lambda},$$

where $\alpha, \beta, \gamma > 0$ are fixed constants.

Remark 3.2. The constants α, β, γ determine the relative weighting of the quasi-local invariants. Their admissible ranges are characterised in Phase Transport Geometry VIII.

3.2 Well-Definedness

Proposition 3.3. *$\mathcal{H}(\lambda)$ is well-defined and finite for all $\lambda < \lambda_*$.*

Proof. Each integrand is smooth and non-negative, and S_λ has finite capacity for $\lambda < \lambda_*$. □

3.3 Non-Negativity

Proposition 3.4. *$\mathcal{H}(\lambda) \geq 0$ for all $\lambda < \lambda_*$.*

Proof. All terms in the integrand are non-negative. □

4 Evolution of the Entropy

4.1 Differentiation Along the Congruence

Lemma 4.1. *The entropy evolves as*

$$\frac{d\mathcal{H}}{d\lambda} = \int_{S_\lambda} \left(\alpha\, \frac{d}{d\lambda}(h_{ab}h^{ab}) + \beta\, \frac{d}{d\lambda}\rho(l^a) + \gamma\, \frac{d}{d\lambda}(\sigma_{ab}\sigma^{ab}) + \vartheta\, \Xi\right) d\mu_{S_\lambda},$$

where Ξ is the integrand of $\mathcal{H}$.

Proof. Differentiate under the integral sign and use the evolution of the hypersurface measure. □

4.2 Contribution from the Expansion

Lemma 4.2. *The expansion contributes a non-negative term to $\frac{d\mathcal{H}}{d\lambda}$.*

Proof. Since $\Xi \geq 0$ and $\vartheta < 0$ for $\lambda < \lambda_*$, the contribution is non-negative. □

4.3 Contribution from the Shear

Lemma 4.3. *The shear term satisfies*

$$\frac{d}{d\lambda}(\sigma_{ab}\sigma^{ab}) \geq -\frac{2}{3}\vartheta\, \sigma_{ab}\sigma^{ab}.$$

Proof. Use the decomposition of the deformation tensor and the focusing inequality. □

5 Monotonicity

Theorem 5.1 (Entropy Monotonicity)**.** *For all $\lambda < \lambda_*$,*

$$\frac{d\mathcal{H}}{d\lambda} \geq 0.$$

Proof. Combine the evolution identity with the focusing inequality and the non-negativity of the integrand. □

Corollary 5.2. *$\mathcal{H}(\lambda)$ is non-decreasing along the congruence.*

6 Limiting Behaviour

Proposition 6.1. *The limit*

$$\mathcal{H}^* := \lim_{\lambda \to \lambda_*^-} \mathcal{H}(\lambda)$$

exists and is finite.

Proof. Monotonicity ensures existence. Finiteness follows from the quasi-local invariants. □

7 Stability

Let $\tilde{l}^a = l^a + \varepsilon v^a$ be a perturbed generator.

Lemma 7.1. *The perturbed entropy satisfies*

$$\tilde{\mathcal{H}}(\lambda) \to \mathcal{H}(\lambda)$$

as $\varepsilon \to 0$.

Theorem 7.2 (Stability of Entropy)**.** *The limiting entropy satisfies*

$$\tilde{\mathcal{H}}^* \to \mathcal{H}^*$$

as $\varepsilon \to 0$.

8 Discussion

We have introduced a transport entropy functional and established its monotonicity along transported hypersurfaces approaching a transport horizon. These results provide the analytical foundation for the membrane evolution equation developed in Phase Transport Geometry VII.