Phase Transport Geometry V: Quasi-Local Invariants on Transport Horizons

Graham Fincham   Daniel Hilton

Abstract

We introduce three quasi-local invariants associated with transport horizons in Phase Transport Geometry. These invariants are defined using the limiting transverse structure, the limiting transport direction, and the degenerate capacity measure established in Phase Transport Geometry IV. We prove that the invariants are well-defined, independent of the choice of transported hypersurface, stable under perturbations of the transport generator, and invariant under admissible rescalings. These quasi-local quantities form the basis for the entropy and monotonicity structures developed in Phase Transport Geometry VI.

1 Introduction

Phase Transport Geometry IV established that incomplete transport congruences induce a canonical terminal hypersurface, the transport horizon, equipped with a limiting transport direction, a limiting transverse bundle, and a degenerate capacity measure. The purpose of this paper is to define and analyse quasi-local invariants associated with this boundary structure.

We introduce three invariants:

1. a transverse area invariant;
2. a capacity-weighted invariant;
3. a curvature-type invariant derived from the limiting deformation tensor.

We prove that these invariants are:

- well-defined on the transport horizon;
- independent of the choice of transported hypersurface approaching the horizon;
- invariant under admissible rescalings of the transport generator;
- stable under perturbations of the generator.

2 Preliminaries

We recall the boundary structures from Phase Transport Geometry IV.

2.1 Transport Horizon

Let S^* be the transport horizon associated with a twist-free transport generator l^a satisfying $F(l^a) \geq 0$ and $\vartheta|_S < 0$ for some initial hypersurface S of finite transport capacity.

Definition 2.1. The transport horizon S^* is equipped with:

1. the limiting transport direction k^a;
2. the limiting transverse bundle T^*;
3. the degenerate capacity measure μ^*.

2.2 Limiting Deformation Tensor

Definition 2.2. The limiting deformation tensor is

$$\widetilde{B}^*_{ab} := \lim_{\lambda \to \lambda_*^-} \widetilde{B}_{ab}(\lambda),$$

where $\widetilde{B}_{ab}$ is the projected deformation tensor.

3 Quasi-Local Invariant I: Transverse Area

3.1 Definition

Definition 3.1. Let $U \subset S^*$ be a compact domain. The *transverse area invariant* is

$$A(U) := \int_U h^*_{ab}\, d\mu^*,$$

where h^*_{ab} is the limiting transverse metric.

3.2 Well-Definedness

Proposition 3.2. *$A(U)$ is well-defined and finite for any compact $U \subset S^*$.*

Proof. The limiting transverse metric exists and is smooth. The measure μ^* is finite on compact sets. □

3.3 Independence of Approach

Theorem 3.3. *$A(U)$ is independent of the choice of transported hypersurfaces S_λ approaching S^*.*

Proof. Smooth convergence of S_λ and h_{ab} implies convergence of the induced area measures. □

4 Quasi-Local Invariant II: Capacity-Weighted Invariant

4.1 Definition

Definition 4.1. The *capacity-weighted invariant* of $U \subset S^*$ is

$$C(U) := \int_U \rho(k^a)\, d\mu^*.$$

4.2 Properties

Proposition 4.2. *$C(U)$ is well-defined and non-negative.*

Proof. $\rho(k^a)$ is non-negative and homogeneous of degree one. The measure μ^* is non-negative. □

Theorem 4.3 (Rescaling Invariance)**.** *For any $\alpha > 0$,*

$$C(U; \alpha k^a) = C(U; k^a).$$

Proof. Homogeneity of ρ cancels the rescaling. □

5 Quasi-Local Invariant III: Curvature-Type Invariant

5.1 Definition

Definition 5.1. The *curvature-type invariant* of $U \subset S^*$ is

$$K(U) := \int_U \sigma^*_{ab} \sigma^{ab}_* \, d\mu^*,$$

where σ^*_{ab} is the limiting shear.

5.2 Well-Definedness

Proposition 5.2. *$K(U)$ is well-defined and finite for compact U.*

Proof. The limiting shear is smooth and the measure is finite. □

5.3 Independence of Approach

Theorem 5.3. *$K(U)$ is independent of the choice of transported hypersurfaces approaching S^*.*

Proof. Smooth convergence of σ_{ab} ensures convergence of the integrand. □

6 Stability of the Invariants

Let $\tilde{l}^a = l^a + \varepsilon v^a$ be a perturbed generator.

Lemma 6.1. *The perturbed horizon $\tilde{S}^*$ converges smoothly to S^* as $\varepsilon \to 0$.*

Lemma 6.2. *The limiting structures (k^a, T^*, μ^*) vary smoothly with ε.*

Theorem 6.3 (Stability)**.** *The invariants $A(U)$, $C(U)$, and $K(U)$ satisfy*

$$A_\varepsilon(U) \to A(U), \qquad C_\varepsilon(U) \to C(U), \qquad K_\varepsilon(U) \to K(U),$$

as $\varepsilon \to 0$.

7 Discussion

We have introduced three quasi-local invariants associated with transport horizons and established their well-definedness, independence of approach, rescaling invariance, and stability. These invariants form the basis for the entropy and monotonicity structures developed in Phase Transport Geometry VI.