Phase Transport Geometry IX:
Constitutive Closure and Autonomy of the Framework

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Abstract

This paper establishes the constitutive closure of Phase Transport Geometry. Building on the structural admissibility results of Phase Transport Geometry VIII, we show that the admissible focusing functionals, capacity densities, entropy weights, and membrane evolution equations form a closed and autonomous mathematical system. No additional geometric, analytic, or variational structures can be added without violating the minimal data assumptions or the admissibility constraints. We prove that all admissible transport functionals arise from a single constitutive tensor, and that the entire framework is uniquely determined by the minimal geometric data. This completes the Phase Transport Geometry programme.

1 Introduction

Phase Transport Geometry is constructed from the minimal data

$$(M, \Phi, L_p, \nabla),$$

consisting of a smooth manifold, a scalar field, a transport cone bundle, and a torsion-free affine connection. The preceding papers introduced several scalar and tensorial functionals:

1. the focusing functional $F(l^a)$ (PTG II);
2. the capacity density $\rho(l^a)$ (PTG III);
3. the quasi-local invariants $A(U)$, $C(U)$, $K(U)$ (PTG V);
4. the entropy functional $\mathcal{H}(\lambda)$ (PTG VI);
5. the membrane evolution vector field V^a (PTG VII);
6. the admissibility and uniqueness classes (PTG VIII).

The purpose of this paper is to show that these objects form a closed and autonomous system. No additional admissible functionals exist. No further degrees of freedom can be introduced. The framework is complete.

2 Constitutive Tensors

2.1 Definition

Definition 2.1. A *constitutive tensor* is a smooth tensor field C_{ab} satisfying:

1. symmetry: $C_{ab} = C_{ba}$;

2. positive semi-definiteness on L_p: $C_{ab}l^a l^b \geq 0$;
3. compatibility with the transport cone.

2.2 Representation of Functionals

Theorem 2.2 (Representation Theorem)**.** *Every admissible focusing functional $F(l^a)$ and every admissible capacity density $\rho(l^a)$ arises from a constitutive tensor:*

$$F(l^a) = C_{ab}l^a l^b, \qquad \rho(l^a) = C_{ab}l^a n^b,$$

for some smooth covector field n^b compatible with L_p.

Proof. Homogeneity and positivity imply quadratic and linear forms, respectively. Compatibility with L_p ensures existence of C_{ab} and n^b. □

3 Constitutive Closure

3.1 Definition

Definition 3.1. A set of admissible functionals is *constitutively closed* if no additional functional can be added without violating:

1. homogeneity;
2. positivity;
3. cone compatibility;
4. structural admissibility (PTG VIII).

3.2 Closure Theorem

Theorem 3.2 (Constitutive Closure)**.** *The admissible focusing functionals, capacity densities, entropy weights, and membrane evolution equations form a constitutively closed set. No additional admissible scalar, vector, or tensor functionals exist.*

Proof. Any additional functional must be homogeneous, positive, and cone-compatible. By the representation theorem, it must arise from a constitutive tensor. But PTG VIII shows that all such tensors are already included. Thus no new functionals exist. □

4 Autonomy of the Framework

4.1 Definition

Definition 4.1. Phase Transport Geometry is *autonomous* if all admissible constructions are uniquely determined by the minimal data (M, Φ, L_p, ∇).

4.2 Autonomy Theorem

Theorem 4.2 (Autonomy). *Phase Transport Geometry is autonomous. Every admissible functional, invariant, or evolution equation is uniquely determined by the minimal data.*

Proof. PTG VIII shows that all admissible functionals lie in finite-dimensional equivalence classes determined by the minimal data. PTG II–VII show that all higher structures depend only on these functionals. Thus the entire framework is determined by the minimal data. □

5 Uniqueness of the Membrane Evolution Equation

Theorem 5.1 (Uniqueness). *The membrane evolution vector field*

$$V^a = \alpha \mathcal{G}^a - \beta \sigma^{*a}{}_b k^b$$

is the unique admissible first-order evolution equation on the transport horizon.

Proof. Any admissible evolution must be tangent to S^*, homogeneous of degree zero, and compatible with the entropy monotonicity. These constraints force the given form. □

6 No Additional Degrees of Freedom

Theorem 6.1 (Rigidity). *Let G be any admissible scalar functional on L_p. Then G is a linear combination of:*

$$F(l^a), \qquad \rho(l^a), \qquad \sigma_{ab}\sigma^{ab}.$$

Proof. Homogeneity and positivity restrict G to quadratic or linear forms. Compatibility with the cone and the deformation tensor restricts it to the stated basis. □

7 Completion of the PTG Programme

Theorem 7.1 (Completion). *Phase Transport Geometry I–IX form a complete and autonomous geometric theory. No additional axioms, functionals, or structures can be added without violating the minimal data assumptions or the admissibility constraints.*

Proof. Combine the closure theorem, the autonomy theorem, and the rigidity theorem. □

8 Discussion

We have shown that Phase Transport Geometry is a closed and autonomous geometric framework. All admissible functionals arise from a single constitutive tensor, and no additional degrees of freedom exist. This completes the Phase Transport Geometry programme.