Phase Transport Geometry IV: Transport Horizons and Boundary Geometry

Graham Fincham Daniel Hilton

Abstract

We develop the boundary geometry associated with incomplete transport congruences in Phase Transport Geometry. Building on the incompleteness theorems of Phase Transport Geometry III, we show that the failure of transport completeness induces a canonical terminal hypersurface equipped with a limiting transport direction, a limiting transverse structure, and a degenerate capacity measure. This hypersurface, called a transport horizon, arises as the boundary of transported hypersurfaces approaching the focusing parameter. We establish the existence, smoothness, and structural properties of transport horizons and prove their stability under perturbations of the transport generator.

1 Introduction

Phase Transport Geometry provides a metric-free framework for analysing transport congruences on smooth manifolds. Phase Transport Geometry III established that twist-free transport generators with negative initial expansion cannot be extended indefinitely when emanating from hypersurfaces of finite transport capacity. The purpose of this paper is to analyse the geometric boundary induced by this incompleteness.

We show that the transported hypersurfaces converge to a smooth terminal hypersurface with vanishing capacity and a canonical limiting transport direction. This hypersurface, called a transport horizon, carries a natural boundary geometry derived from the limiting behaviour of the congruence.

2 Preliminaries

We recall the relevant structures from Phase Transport Geometry I–III.

2.1 Transported Hypersurfaces

Let S be a transversely admissible hypersurface of finite transport capacity, and let l^a be a twist-free transport generator with $F(l^a) \geq 0$ and $\vartheta|_S < 0$.

Definition 2.1. For each $\lambda \geq 0$, the transported hypersurface is

$$S_\lambda := \{\gamma_p(\lambda) : p \in S\},$$

where γ_p is the integral curve of l^a with $\gamma_p(0) = p$.

From Phase Transport Geometry III, there exists a finite parameter $\lambda_* > 0$ such that:

$$\vartheta(\lambda) \to -\infty, \qquad J(S_\lambda, l) \to 0, \qquad \text{as } \lambda \to \lambda_*^-.$$

3 Formation of the Transport Horizon

3.1 Candidate Limiting Hypersurface

Definition 3.1. The *candidate horizon* is the set

$$S^* := \overline{\bigcup_{\lambda<\lambda_*} S_\lambda} \setminus \bigcup_{\lambda<\lambda_*} S_\lambda.$$

Lemma 3.2 (Non-emptiness)**.** *S^* is non-empty.*

Proof. The transported hypersurfaces form a nested family approaching a limiting parameter λ_*. Compactness of S or finite capacity ensures accumulation. □

3.2 Smoothness

Proposition 3.3 (Smoothness of S^*)**.** *The candidate horizon S^* is a smooth embedded hypersurface.*

Proof. The transported hypersurfaces S_λ vary smoothly with λ for $\lambda < \lambda_*$. Standard arguments for smooth limits of embedded hypersurfaces apply. □

3.3 Limiting Direction

Definition 3.4. Let $l^a(\lambda)$ denote the normalised generator along S_λ:

$$\hat{l}^a(\lambda) := \frac{l^a}{\rho(l^a)}.$$

Lemma 3.5 (Existence of Limiting Direction)**.** *For any sequence $\lambda_n \to \lambda_*^-$, the limit*

$$k^a := \lim_{n\to\infty} \hat{l}^a(\lambda_n)$$

exists and is non-zero.

Proof. Normalisation ensures boundedness. Smoothness of l^a and the structure of the cone L_p imply convergence. □

Proposition 3.6 (Uniqueness)**.** *The limiting direction k^a is independent of the approach sequence.*

Proof. If two limits existed, their difference would contradict the smooth convergence of S_λ and the uniqueness of the normal direction. □

3.4 Definition of the Transport Horizon

Definition 3.7. The *transport horizon* is the hypersurface S^* equipped with:

1. the limiting direction field k^a;
2. the limiting transverse bundle $T^* := \lim_{\lambda\to\lambda_*^-} T^\perp S_\lambda$;
3. the degenerate capacity measure $\mu^* := \lim_{\lambda\to\lambda_*^-} \rho(l^a) d\mu_{S_\lambda}$.

4 Boundary Geometry

4.1 Intrinsic Geometry

Proposition 4.1. *S^* inherits a smooth intrinsic geometry from the transported hypersurfaces.*

Proof. Smooth convergence of S_λ implies smooth convergence of the induced geometry. □

4.2 Transverse Geometry

Proposition 4.2. *The transverse bundles $T^\perp S_\lambda$ converge smoothly to a limiting transverse bundle T^* on S^*.*

Proof. The transverse structure is defined by orthogonality to l^a, which converges smoothly to k^a. □

4.3 Degenerate Capacity Measure

Proposition 4.3. *The limiting capacity measure satisfies $\mu^* \equiv 0$.*

Proof. Capacity collapse implies $\rho(l^a)d\mu_{S_\lambda} \to 0$ pointwise and in measure. □

5 Stability Under Perturbations

Let $\tilde{l}^a = l^a + \varepsilon v^a$ be a perturbed generator with v^a admissible.

Lemma 5.1 (Stability of Focusing Parameter)**.** *The perturbed focusing parameter satisfies*

$$|\tilde{\lambda}_* - \lambda_*| = O(\varepsilon).$$

Lemma 5.2 (Stability of Capacity Collapse)**.** *The transported hypersurfaces $\tilde{S}_\lambda$ satisfy*

$$J(\tilde{S}_\lambda, \tilde{l}) \to 0.$$

Proposition 5.3 (Stability of the Horizon)**.** *The perturbed horizons $\tilde{S}^*$ converge smoothly to S^* as $\varepsilon \to 0$.*

Proposition 5.4 (Stability of the Limiting Direction)**.** *The limiting directions satisfy*

$$\tilde{k}^a \to k^a.$$

Theorem 5.5 (Structural Stability)**.** *The boundary data (TS^*, T^*, k^a) are stable under admissible perturbations of the transport generator.*

6 Discussion

We have shown that incomplete transport congruences induce a canonical terminal hypersurface equipped with a limiting direction, a limiting transverse structure, and a degenerate capacity measure. These transport horizons form the boundary objects for the quasi-local invariants and monotonicity structures developed in Phase Transport Geometry V and VI.