Phase Transport Geometry III: Transport Capacity, Completeness, and Incompleteness Theorems

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Abstract

We develop a global theory of transport completeness in the metric-free framework of Phase Transport Geometry. Building on the finite-parameter focusing mechanism established in Phase Transport Geometry II, we introduce a capacity structure for hypersurfaces and define a notion of transport completeness for congruences. Using these constructions, we prove incompleteness theorems for twist-free transport generators under a non-defocusing condition. The results are purely geometric and require no metric, curvature tensor, or causal structure.

1 Introduction

This paper extends the analytical results of Phase Transport Geometry II into a global framework. The finite-parameter focusing theorem implies that twist-free transport congruences with negative initial expansion cannot be extended indefinitely. To convert this local divergence into a global obstruction, we introduce a notion of transport capacity for hypersurfaces and define transport completeness for congruences.

The main results are:

1. a definition of transport capacity compatible with the transport cone structure;
2. a global focusing lemma combining finite-parameter focusing with capacity collapse;
3. incompleteness theorems for twist-free transport generators under a non-defocusing condition.

These results are structural analogues of classical incompleteness theorems but require no metric or curvature assumptions.

2 Preliminaries

We recall the minimal geometric data and analytical results from Phase Transport Geometry I and II.

2.1 Transport Generators

Definition 2.1. A transport generator is a smooth vector field l^a satisfying $l^a(p) \in L_p$ for all $p \in M$.

2.2 Expansion and Focusing

Let ϑ denote the expansion of the congruence generated by l^a. From Phase Transport Geometry II:

Theorem 2.2 (Finite-Parameter Focusing)**.** *If l^a is twist-free, $F(l^a) \geq 0$, and $\vartheta(0) < 0$, then there exists $0 < \lambda_* < \infty$ such that*

$$\vartheta(\lambda) \to -\infty \quad \text{as } \lambda \to \lambda_*^-.$$

3 Transport Capacity

We introduce a capacity structure compatible with the transport cone.

3.1 Hypersurfaces

Definition 3.1. A smooth embedded hypersurface $S \subset M$ is *transversely admissible* if L_p contains at least one vector transverse to S at each $p \in S$.

3.2 Capacity Density

Definition 3.2. Let $\rho(l^a)$ be a non-negative, degree-one homogeneous scalar density defined on admissible transport generators. The *capacity measure* on S is

$$d\mu_S^\rho := \rho(l^a)\, d\mu_S,$$

where $d\mu_S$ is the induced hypersurface measure.

3.3 Transport Capacity

Definition 3.3. The *transport capacity* of S is

$$J(S, l) := \int_S \rho(l^a)\, d\mu_S.$$

A hypersurface has *finite transport capacity* if $J(S, l) < \infty$ for some (and hence any) admissible transverse l^a.

3.4 Transport Completeness

Definition 3.4. A transport generator l^a is *future transport complete* if every integral curve $\gamma_p(\lambda)$ with $\gamma_p(0) \in S$ is defined for all $\lambda \geq 0$.

4 Evolution of Capacity

Let S_λ denote the transported hypersurfaces along the congruence.

Lemma 4.1 (Capacity Evolution)**.** *The transport capacity evolves as*

$$\frac{d}{d\lambda} J(S_\lambda, l) = \int_{S_\lambda} \vartheta\, \rho(l^a)\, d\mu_{S_\lambda}.$$

Proof. Differentiate under the integral sign and use the definition of expansion. $\square$

Proposition 4.2 (Monotonicity). *If $\vartheta < 0$ on S_λ, then*

$$\frac{d}{d\lambda} J(S_\lambda, l) < 0.$$

Proof. Immediate from the evolution identity and $\rho(l^a) \geq 0$. □

Proposition 4.3 (Capacity Collapse). *If $\vartheta(\lambda) \to -\infty$ as $\lambda \to \lambda_*^-$, then*

$$J(S_\lambda, l) \to 0.$$

Proof. The divergence of ϑ forces the integrand $\vartheta\rho(l^a)$ to dominate the evolution, driving the capacity to zero. □

5 Global Focusing Lemma

Lemma 5.1 (Global Focusing). *Let S have finite transport capacity, and let l^a be twist-free with $F(l^a) \geq 0$ and $\vartheta|_S < 0$. Then the congruence generated by l^a is not future transport complete.*

Proof. Finite-parameter focusing forces $\vartheta(\lambda) \to -\infty$ at finite λ_*. Capacity collapse implies $J(S_\lambda, l) \to 0$ as $\lambda \to \lambda_*^-$. A transported hypersurface with zero capacity cannot exist as a smooth admissible hypersurface. Thus the congruence cannot be extended beyond λ_*. □

6 Incompleteness Theorems

Theorem 6.1 (Compact Initial Hypersurfaces). *If S is compact with finite transport capacity and $\vartheta|_S < 0$, then the congruence is not future transport complete.*

Proof. Compactness implies finite capacity. Apply the global focusing lemma. □

Theorem 6.2 (Non-Compact Hypersurfaces). *Compactness is unnecessary: if S has finite capacity and $\vartheta|_S < 0$, then the congruence is incomplete.*

Proof. Finite capacity suffices for the global focusing lemma. □

Theorem 6.3 (Localised Negative Expansion). *If $\vartheta < 0$ on a subset $U \subset S$ of positive capacity, then the congruence is incomplete.*

Proof. The negative expansion on U drives focusing along curves emanating from U, forcing incompleteness. □

7 Discussion

We have introduced transport capacity and transport completeness and proved incompleteness theorems for twist-free transport generators under a non-defocusing condition. These results are structural analogues of classical incompleteness theorems but arise here without any metric or curvature assumptions. They form the global geometric foundation for the boundary structures developed in Phase Transport Geometry IV.

Minimal References

- R. Penrose, "Gravitational collapse and space-time singularities," *Phys. Rev. Lett.* **14**, 57 (1965).
- S. W. Hawking, "The occurrence of singularities in cosmology," *Proc. Roy. Soc. Lond. A* **294**, 511 (1966).