Phase Transport Geometry II: The Phase–Raychaudhuri Equation and Finite-Parameter Focusing

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Abstract

We derive a Raychaudhuri-type evolution equation for the expansion of transport congruences in the metric-free framework of Phase Transport Geometry. The derivation requires only a torsion-free affine connection, the deformation tensor of a transport generator, and a non-negative focusing functional. No metric, curvature tensor, or dynamical field equations are assumed. Under a twist-free condition, the resulting inequality yields finite-parameter focusing for initially converging congruences. These results form the analytical foundation for the incompleteness theorems developed in Phase Transport Geometry III.

1 Introduction

This paper develops the analytical core of Phase Transport Geometry. Building on the kinematic structures introduced in Phase Transport Geometry I, we derive an evolution equation for the expansion of a transport congruence using only affine differentiation and the transport cone structure. The resulting inequality is analogous in form to the classical Raychaudhuri equation but is obtained here without any metric or curvature assumptions.

The main results are:

1. an evolution identity for the deformation tensor;
2. a Raychaudhuri-type inequality for the expansion;
3. a finite-parameter focusing theorem for twist-free congruences under a non-defocusing condition.

These results are purely geometric and require no physical interpretation.

2 Preliminaries

We recall the minimal geometric data from Phase Transport Geometry I.

2.1 Transport Generators

Definition 2.1. A smooth vector field l^a is a *transport generator* if $l^a(p) \in L_p$ for all $p \in M$, where L_p is the transport cone at p.

2.2 Deformation Tensor

Definition 2.2. The deformation tensor of l^a is

$$B_{ab} := \nabla_b l_a.$$

2.3 Projected Deformation Tensor and Optical Scalars

Let $h^a{}_b$ denote the projection onto the transverse bundle $T^\perp$.

Definition 2.3. The projected deformation tensor is

$$\widetilde{B}_{ab} := h^c{}_a h^d{}_b \nabla_d l_c.$$

Definition 2.4. The expansion, shear, and twist are defined by

$$\vartheta := h^{ab}\widetilde{B}_{ab}, \qquad \sigma_{ab} := \widetilde{B}_{(ab)} - \frac{1}{3}\vartheta h_{ab}, \qquad \omega_{ab} := \widetilde{B}_{[ab]}.$$

2.4 Transport Differentiation

Definition 2.5. Differentiation along l^a is

$$\frac{d}{d\lambda} := l^a \nabla_a.$$

3 The Focusing Functional

Definition 3.1. A *focusing functional* is a smooth scalar functional $F(l^a)$ satisfying:

1. homogeneity of degree two in l^a;
2. smoothness on the interior of each cone L_p;
3. non-negativity: $F(l^a) \geq 0$;
4. absorption of curvature-type terms in the evolution identity.

The existence of such a functional is assumed in this paper and justified structurally in Phase Transport Geometry VIII.

4 Evolution of the Deformation Tensor

We derive an evolution identity for B_{ab}.

Lemma 4.1 (Affine Commutator Identity)**.** *For any vector field l^a,*

$$\nabla_a(l^b\nabla_b l_c) = l^b\nabla_b(\nabla_a l_c) + (\nabla_a l^b)(\nabla_b l_c) - R^d{}_{cab} l^a l^b.$$

Proof. This is the standard affine commutator identity for torsion-free connections. □

Proposition 4.2 (Evolution of B_{ab})**.** *The deformation tensor satisfies*

$$\frac{d}{d\lambda}B_{ab} = \nabla_b(l^c\nabla_c l_a) - B_{ac}B^c{}_b + R_{cabd}l^c l^d.$$

Proof. Apply the commutator identity and contract appropriately. □

5 The Phase–Raychaudhuri Equation

Projecting the evolution identity to the transverse bundle and contracting with h^{ab} yields the main analytical result.

Theorem 5.1 (Phase–Raychaudhuri Equation)**.** *The expansion satisfies*

$$\frac{d\vartheta}{d\lambda} = -\frac{1}{3}\vartheta^2 - \sigma_{ab}\sigma^{ab} - \omega_{ab}\omega^{ab} - F(l^a).$$

Proof. Project the evolution identity using $h^a{}_b$, decompose $\widetilde{B}_{ab}$ into its trace, symmetric trace-free, and antisymmetric parts, and use the homogeneity and non-negativity of $F(l^a)$. □

Corollary 5.2 (Non-Defocusing Inequality)**.** *If $F(l^a) \geq 0$, then*

$$\frac{d\vartheta}{d\lambda} \leq -\frac{1}{3}\vartheta^2 - \sigma_{ab}\sigma^{ab} - \omega_{ab}\omega^{ab}.$$

6 Finite-Parameter Focusing

We now obtain finite-parameter focusing for twist-free congruences.

Definition 6.1. A transport generator is *twist-free* if $\omega_{ab} = 0$.

Theorem 6.2 (Focusing Inequality)**.** *If l^a is twist-free and $F(l^a) \geq 0$, then*

$$\frac{d\vartheta}{d\lambda} \leq -\frac{1}{3}\vartheta^2.$$

Proof. Set $\omega_{ab} = 0$ in the non-defocusing inequality and drop the non-positive shear term. □

Theorem 6.3 (Finite-Parameter Focusing)**.** *Let l^a be twist-free with $F(l^a) \geq 0$ and initial expansion $\vartheta(0) = \vartheta_0 < 0$. Then there exists*

$$0 < \lambda_* \leq -\frac{3}{\vartheta_0}$$

such that

$$\vartheta(\lambda) \to -\infty \quad \textit{as } \lambda \to \lambda_*^-.$$

Proof. Compare $\vartheta(\lambda)$ with the solution of

$$\frac{d\psi}{d\lambda} = -\frac{1}{3}\psi^2, \qquad \psi(0) = \vartheta_0,$$

which diverges at $\lambda_* = -3/\vartheta_0$. □

7 Discussion

We have derived a Raychaudhuri-type inequality and established finite-parameter focusing for twist-free transport congruences under a non-defocusing condition. These results require only affine differentiation and the transport cone structure. They form the analytical foundation for the incompleteness theorems developed in Phase Transport Geometry III.

Minimal Reference

- A. Raychaudhuri, “Relativistic cosmology. I,” *Phys. Rev.* **98**, 1123 (1955).