Phase Transport Geometry I:
A Metric-Free Transport Framework for Scalar Fields

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Abstract

We introduce Phase Transport Geometry, a metric-free geometric framework constructed from a smooth scalar field, an admissible transport cone bundle, and a torsion-free affine connection. From these minimal ingredients we define transport generators, transport curves, the deformation tensor, a transverse bundle equipped with a positive-definite inner product, and the associated optical scalars. All constructions are purely geometric and require no metric, curvature tensor, or dynamical field equations. This paper establishes the foundational kinematic structures that underpin the subsequent development of transport focusing, incompleteness, boundary geometry, quasi-local invariants, entropy, and membrane dynamics.

1 Introduction

This paper develops the foundational geometric structures of Phase Transport Geometry. The framework is constructed from minimal data: a smooth manifold, a scalar field, a cone of admissible transport directions, and a torsion-free affine connection. No metric, curvature tensor, or physical interpretation is assumed. The purpose of this paper is to define the basic kinematic objects associated with transport: transport generators, transport curves, the deformation tensor, the transverse bundle, and the optical scalars.

The results of this paper are purely local and structural. No focusing inequalities, completeness criteria, boundary structures, or entropy-type quantities are introduced here. These appear in subsequent papers of the series.

Structure of the Paper

Section 2 introduces the minimal geometric data. Section 3 defines transport curves, the deformation tensor, and the transverse bundle. Section 4 introduces the optical scalars and proves the decomposition of the projected deformation tensor. Section 5 summarises the role of these constructions in the broader programme.

2 Minimal Geometric Data

2.1 Underlying Manifold

Definition 2.1. Let M be a smooth, connected, Hausdorff, second-countable 4-manifold. No metric structure is assumed. All constructions are invariant under diffeomorphisms of M.

2.2 Scalar Field and Phase Orientation

Definition 2.2. A smooth scalar field $\Phi : M \to \mathbb{R}$ is fixed. A vector $l^a \in T_pM$ is *phase-oriented* if

$$l^a \nabla_a \Phi \geq 0.$$

2.3 Transport Cone Bundle

Definition 2.3. At each point $p \in M$, let $L_p \subset T_pM$ be a nonempty, closed, convex cone satisfying:

1. Positive scaling: $l^a \in L_p \Rightarrow \alpha l^a \in L_p$ for all $\alpha > 0$.
2. Convexity: if $l_1, l_2 \in L_p$ and $\lambda \in [0, 1]$, then $\lambda l_1 + (1 - \lambda) l_2 \in L_p$.
3. Phase orientation: $l^a \nabla_a \Phi \geq 0$ for all $l^a \in L_p$.

The assignment $p \mapsto L_p$ is upper semicontinuous.

2.4 Transport Generators and Curves

Definition 2.4. A smooth vector field l^a is a *transport generator* if $l^a(p) \in L_p$ for all $p \in M$. A C^1 curve $\gamma : I \to M$ is a *transport curve* if $\dot{\gamma}^a(\lambda) \in L_{\gamma(\lambda)}$ for all $\lambda \in I$.

2.5 Affine Connection

Definition 2.5. The manifold is equipped with a torsion-free affine connection ∇, used solely for covariant differentiation.

2.6 Transport Differentiation

Definition 2.6. Differentiation along a transport generator is defined by

$$\frac{d}{d\lambda} := l^a \nabla_a.$$

3 Deformation Tensor and Transverse Geometry

3.1 Deformation Tensor

Definition 3.1. The *deformation tensor* of a transport generator l^a is

$$B_{ab} := \nabla_b l_a.$$

3.2 Transverse Bundle

Definition 3.2. The *transverse bundle* at $p \in M$ is

$$T_p^\perp := \{X^a \in T_pM : X^a l_a = 0\}.$$

Definition 3.3. Let $h^a{}_b$ denote the projection onto $T^\perp$. A positive-definite inner product h_{ab} is fixed on $T^\perp$.

3.3 Projected Deformation Tensor

Definition 3.4. The *projected deformation tensor* is

$$\widetilde{B}_{ab} := h^c{}_a h^d{}_b \nabla_d l_c.$$

4 Optical Scalars and Decomposition

4.1 Expansion

Definition 4.1. The *expansion* is

$$\vartheta := h^{ab}\widetilde{B}_{ab}.$$

4.2 Shear and Twist

Definition 4.2. The *shear* is

$$\sigma_{ab} := \widetilde{B}_{ab} - \frac{1}{3}\vartheta h_{ab}.$$

Definition 4.3. The *twist* is the antisymmetric part

$$\omega_{ab} := \widetilde{B}_{[ab]}.$$

4.3 Decomposition Theorem

Theorem 4.4 (Optical Decomposition)**.** *The projected deformation tensor decomposes as*

$$\widetilde{B}_{ab} = \frac{1}{3}\vartheta h_{ab} + \sigma_{ab} + \omega_{ab}.$$

Proof. The projection $h^a{}_b$ induces an orthogonal decomposition of $T^\perp \otimes T^\perp$ into trace, symmetric trace-free, and antisymmetric components. The result follows by applying this decomposition to $\widetilde{B}_{ab}$. □

5 Discussion

This paper establishes the foundational kinematic structures of Phase Transport Geometry. Beginning with only a scalar field, a transport cone bundle, and a torsion-free affine connection, we have defined transport generators, transport curves, the deformation tensor, the transverse bundle, and the optical scalars. These structures form the basis for the analytical, global, and boundary results developed in subsequent papers of the series.